Published by Institute of Physics Publishing for SISSA

Received: *April 3, 2006*
Revised: *December 20, 2006*
Accepted: *January 11, 2007*
Published: *February 5, 2007*

q-Deformed SU(2) instantons by q-quaternions

Gaetano Fiore

Dip. di Matematica e Applicazioni, Università "Federico II"
v. Claudio 21, 80125 Napoli, Italy, and
I.N.F.N., Sezione di Napoli, Complesso MSA,
V. Cintia, 80126 Napoli, Italy
E-mail: gaetano.fiore@na.infn.it

Abstract: Interpreting the coordinates of the quantum Euclidean space $\mathbb{R}_q^4$ [the $SO_q(4)$-covariant noncommutative space] as the entries of a "q-quaternion matrix" we construct (anti)instanton solutions of a would-be q-deformed $su(2)$ Yang-Mills theory on $\mathbb{R}_q^4$. Since the (anti)selfduality equations are covariant under the quantum group of deformed rotations, translations and scale change, by applying the latter we can respectively generate "gauge equivalent" or "inequivalent" solutions from the one centered at the origin and with unit size. We also construct multi-instanton solutions. As these solutions depend on noncommuting parameters playing the roles of 'sizes' and 'coordinates of the centers' of the instantons, this indicates that the moduli space of a complete theory should be a noncommutative manifold. Similarly, as the (global) gauge transformations relating "gauge equivalent" solutions depend on the generators of two copies of $SU_q(2)$, this suggests that gauge transformations should be allowed to depend on additional noncommutative parameters.

Keywords: Non-Commutative Geometry, Gauge Symmetry, Solitons Monopoles and Instantons, Quantum Groups.

Contents

1. Introduction

A broad attention has been devoted in recent years to the construction of gauge field theories on noncommutative manifolds. A crucial test of this construction is the search of instantonic solutions, especially after the discovery [33] that deforming $\mathbb{R}^4$ into the Moyal-Weyl noncommutative Euclidean space $\mathbb{R}^4_\theta$ regularizes the zero-size singularities of the instanton moduli space (see also [41]). Various other noncommutative geometries have been considered (see e.g. [10, 5, 11, 28]). They do not always completely fit Connes' standard framework of noncommutative geometry [8], thus stimulating attempts of generalizations. Among the available deformations of $\mathbb{R}^4$ there is also the Faddeev-Reshetikhin-Takhtadjan noncommutative Euclidean space $\mathbb{R}^4_q$ covariant under $SO_q(4)$ [14]. This, as other quantum group covariant noncommutative spaces (shortly: quantum spaces), is maybe even more problematic for the formulation [26] of a gauge field theory. One main reason is the lack of a proper (i.e. cyclic) trace to define gauge invariant observables (action, etc). Another one is the complicated $\star$-structure of the differential calculus for real q. Here, leaving these two issues aside, we formulate and solve the (anti)selfduality equations on it; we omit mathematical details and proofs, which can be found in the longer paper [21]. This might contribute to suggest more general formulations of gauge theories on noncommutative manifolds (include quantum spaces) where e.g. gauge transformations, gauge potentials, and the corresponding field strengths depend not only on coordinates, but also on derivatives (as suggested e.g. in [12, 3]) and/or possibly on additional noncommuting parameters (see section 6 below).

As known, the search and classification [2] of Yang-Mills instantons on $\mathbb{R}^4$ is largely simplified when the latter is promoted to the quaternion algebra $\mathbb{H}$. Following the undeformed case, we introduce (section 2) a notion of a q-quaternion as the defining matrix of a copy of $\mathbb{H}_q := SU_q(2) \times \mathbb{R}^{\geq}$ ($\mathbb{R}^{\geq}$ denoting the semigroup of nonnegative real numbers) and reformulate the algebra $\mathcal{A}$ of functions on $\mathbb{R}^4_q$ as a $\star$-bialgebra $C(\mathbb{H}_q)$. The bialgebra structure encodes the property that the product of two quaternions is a quaternion and is inherited from the bialgebra of 2×2 quantum matrices [13, 15, 46, 14] (therefore it differs from the proposal in [31]). It also turns out that the quantum sphere S^4_q of [11] can be regarded as a compactification of this $\star$-algebra. In section 3 we reformulate in q-quaternion language the $SO_q(4)$-covariant differential calculus, the $SO_q(4)$-covariant q-epsilon tensor and Hodge map [17, 18] on $\Omega^*(\mathbb{R}^4_q)$. In section 5 we formulate (anti)selfduality equations and find a large family of solutions A in the form of 1-form valued 2×2 matrices both in the "regular" and in the "singular gauge". There is a larger indeterminacy than in the undeformed theory because we are not yet able to formulate and impose the correct antihermiticity condition on the gauge potential. Among the solutions there are some distinguished choices that closely resemble (in q-quaternion language) their undeformed counterparts (instantons and anti-instantons) in $su(2)$ Yang-Mills theory on $\mathbb{R}^4$. [The (still missing) complete gauge theory might however be a deformed $u(2)$ rather than $su(2)$ Yang-Mills theory.] The projector characterizing the instanton projective module (playing the role of the vector bundle) of [11] in q-quaternion language takes exactly the same natural form as in the undeformed theory. We also point out where the present model doesn't fit the today standard formulation [8] of gauge theory on noncommutative spaces (some basic notions of which we recall in section 4). In analogy with the undeformed (and the Nekrasov-Schwarz [33]) case, applying (section 6) the quantum group $SO_q(4)$ of q-deformed rotations one obtains gauge equivalent solutions (by a global gauge transformation), whereas applying q-deformed dilatations and (the braided group of) q-deformed translations one finds gauge inequivalent solutions; however this global gauge transformation depends on new noncommuting parameters playing the role of coordinates of $SO_q(4)$, and the gauge inequivalent solutions depend on the noncommuting "coordinates of the center" of the (anti)instanton. Finally (section 7), we find first n-instantons solutions in the "singular" gauge for any integer n; the construction procedure is not yet the deformed analog of the general ADHM one [2], but rather of the procedure initiated in [44] and developed in [48], which reduces to the determination of a suitable harmonic scalar potential, expressed in quaternion language. Then for $n = 1, 2$ we transform the singular solutions into "regular" solutions by "singular gauge transformations", as in the undeformed case (of course the $n = 1$ regular instanton solution is again one found in section 5). The solutions are parametrized by noncommuting parameters playing the role of "sizes" and "coordinates of the centers" of the (anti)instantons. This indicates that the moduli space of a complete theory will be a noncommutative manifold. This is similar to what was proposed in [24] for $\mathbb{R}^4_\theta$ for selfdual deformation parameters $\theta_{\mu\nu}$.

2. The q-quaternion bialgebra $C(\mathbb{H}_q)$

We start by recalling how the (undeformed) quaternion $\star$-algebra $\mathbb{H}$ can be formulated in terms of 2×2 matrices: any $X \in \mathbb{H}$ is given by

$$X = \mathrm{x}_1 + \mathrm{x}_2 i + \mathrm{x}_3 j + \mathrm{x}_4 k,$$

with $\mathrm{x} \in \mathbb{R}^4$ and imaginary i, j, k fulfilling

$$i^2 = j^2 = k^2 = -1, \qquad ijk = -1.$$

One refers to x_1 and to the following three terms as to the 'real' and 'imaginary' part of X respectively. Replacing i, j, k by Pauli matrices times the imaginary unit i we can associate to X a matrix

$$X \leftrightarrow x \equiv \begin{pmatrix} \mathrm{x}_1 + \mathrm{x}_4 \mathrm{i} & \mathrm{x}_3 + \mathrm{x}_2 \mathrm{i} \\ -\mathrm{x}_3 + \mathrm{x}_2 \mathrm{i} & \mathrm{x}_1 - \mathrm{x}_4 \mathrm{i} \end{pmatrix} =: \begin{pmatrix} \alpha & -\gamma^\star \\ \gamma & \alpha^\star \end{pmatrix}$$

(where $\alpha, \gamma \in \mathbb{C}$). The quaternionic product becomes represented by matrix multiplication, and the quaternionic conjugation becomes represented by hermitean conjugation of the matrix x. Therefore $\mathbb{H}$ can be seen also as the subalgebra of $M_2(\mathbb{C})$ consisting of all complex 2×2 matrices of this form. Since the determinant of any x is nonnegative,

$$|x|^2 \equiv \det(x) = |a|^2 + |\gamma|^2 \geq 0,$$

any x can be factorized in the form $x = T|x|$, where $T \in \mathrm{SU}(2)$ and $|x|$ belongs to the semigroup $\mathbb{R}^{\geq}$ of nonnegative real numbers. Hence any x belongs also to the semigroup $\mathrm{SU}(2) \times \mathbb{R}^{\geq}$.

We q-deform this just replacing SU(2) by $SU_q(2)$ in the algebra of functions of the matrix elements of x. In other words, we define a q-quaternion just as one introduces the defining matrix of $SU_q(2)$ [45, 46], but without imposing the unit q-determinant condition. For $q \in \mathbb{R}$ consider the unital associative $\star$-algebra $\mathcal{A} \equiv C(\mathbb{H}_q)$ generated by elements $\alpha, \gamma^\star, \alpha^\star, \gamma$ fulfilling the commutation relations

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, & \alpha\gamma^\star &= q\gamma^\star\alpha, & \gamma\alpha^\star &= q\alpha^\star\gamma, \\ \gamma^\star\alpha^\star &= q\alpha^\star\gamma^\star, & [\alpha, \alpha^\star] &= (1-q^2)\gamma\gamma^\star & [\gamma^\star, \gamma] &= 0. \end{aligned} \tag{2.1}$$

Introducing the matrix

$$x \equiv \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} := \begin{pmatrix} \alpha & -q\gamma^\star \\ \gamma & \alpha^\star \end{pmatrix} \tag{2.2}$$

we can rewrite these commutation relations as

$$\hat{R} x_1 x_2 = x_1 x_2 \hat{R} \tag{2.3}$$

and the conjugation relations as $x^{\alpha\beta\star} = \epsilon^{\beta\gamma} x^{\delta\gamma} \epsilon_{\delta\alpha}$, i.e.

$$x^\dagger = \bar{x} \qquad \text{where} \qquad \bar{a} := \epsilon a^T \epsilon^{-1} \quad \forall a \in M_2. \tag{2.4}$$

Here as usual $x_1 \equiv x \otimes_{\mathbb{C}} I_2$, $x_2 \equiv I_2 \otimes_{\mathbb{C}} x$ (I_2 is the 2×2 unit matrix), $\hat{R}$ is the braid matrix of $M_q(2)$, $GL_q(2)$ and $SU_q(2)$

$$\hat{R}^{\alpha\beta}_{\gamma\delta} = q\delta^\alpha_\gamma\delta^\beta_\delta + \epsilon^{\alpha\beta}\epsilon_{\gamma\delta}, \tag{2.5}$$

and ϵ is the corresponding completely antisymmetric tensor

$$\epsilon \equiv (\epsilon_{\alpha\beta}) := \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \qquad \epsilon^{-1} \equiv (\epsilon^{\alpha\beta}) = -q^{-1}(\epsilon_{\alpha\beta}). \tag{2.6}$$

The decomposition of $\hat{R}$ in orthogonal projectors reads

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a, \tag{2.7}$$

and the q-deformed symmetric, antisymmetric projectors $\mathcal{P}_s, \mathcal{P}_a$ can be expressed as

$$\mathcal{P}_a{}^{\alpha\beta}_{\gamma\delta} = -\frac{\epsilon^{\alpha\beta}\epsilon_{\gamma\delta}}{q+q^{-1}}, \qquad \mathcal{P}_s{}^{\alpha\beta}_{\gamma\delta} = \delta^\alpha_\gamma\delta^\beta_\delta + \frac{\epsilon^{\alpha\beta}\epsilon_{\gamma\delta}}{q+q^{-1}}. \tag{2.8}$$

$\mathcal{A} := C(\mathbb{H}_q)$ can be naturally endowed with a $\star$-bialgebra structure (we are not excluding $\mathbf{0}_2$ from the spectrum of x), more precisely the above real section of the bialgebra $C\left(M_q(2)\right)$ of 2×2 quantum matrices [13, 15, 46, 14]. In the sequel we shall write the corresponding coproduct $\Delta(x^{\alpha\gamma}) = x^{\alpha\beta}\otimes x^{\beta\gamma}$ in the more compact matrix product form

$$x \to \Delta(x) = ax \tag{2.9}$$

where we have renamed $x\otimes\mathbf{1} \to a$, $\mathbf{1}\otimes x \to x$. Since the coproduct is a $\star$-algebra map, $\Delta(x)$, or equivalently the matrix product ax of any two matrices a, x with mutually commuting entries and fulfilling (2.3), (2.4), again fulfills the latter. Therefore we shall call any such matrix x a *q-quaternion*, and $\mathcal{A} := C(\mathbb{H}_q)$ the q-quaternion bialgebra. Note that, according to this definition, I_2 is a q-quaternion, and x is a q-quaternion iff $-x$ is. As well-known, the socalled 'q-determinant' of x

$$|x|^2 \equiv \det{}_q(x) := x^{11}x^{22} - qx^{12}x^{21} = \alpha^\star\alpha + \gamma^\star\gamma = \frac{1}{1+q^2}x^{\alpha\alpha'}x^{\beta\beta'}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}, \tag{2.10}$$

is central, manifestly nonnegative-definite and group-like. Therefore in any $\star$-representation it will have zero eigenvalue iff x has $\mathbf{0}_2$ as an eigenvalue matrix. Replacing (2.5) in (2.3) we find that the latter is equivalent to

$$x\bar{x} = \bar{x}x = |x|^2 I_2. \tag{2.11}$$

If we extend $\mathcal{A} = C(\mathbb{H}_q)$ also by the new (central, positive-definite and group-like) generator $|x|^{-1}$ (this will exclude $x = \mathbf{0}_2$ from the spectrum), the matrix x becomes invertible and we obtain even a Hopf $\star$-algebra with antipode S defined by

$$Sx = x^{-1} = \frac{\bar{x}}{|x|^2}, \qquad S|x|^{-1} = |x|. \tag{2.12}$$

The matrix elements of $T := \frac{x}{|x|}$ fulfill the 'RTT' [14] relations (2.3) and

$$T^\dagger = T^{-1} = \overline{T}, \qquad\qquad \det{}_q(T) = \mathbf{1}, \tag{2.13}$$

namely generate $C\left(SU_q(2)\right)$ [45, 46] as a quotient subalgebra. Therefore in this case the $x^{\alpha\alpha'}$ generate the (Hopf) $\star$-algebra $C\left(SU_q(2)\times GL^+(1)\right)$ of functions on the "quantum group $SU_q(2)\times GL^+(1)$ of non-vanishing q-quaternions" [a real section of the Hopf algebra $C\left(GL_q^+(2)\right)$], in analogy with the $q=1$ case.

One can easily verify that as a $\star$-algebra $\mathcal{A} := C(\mathbb{H}_q)$ coincides with the algebra of functions on the $SO_q(4)$-covariant quantum Euclidean Space $\mathbb{R}^4_q$ of [14]. We identify the present $qx^{11}, x^{12}, -qx^{21}, x^{22}$ with the generators x^1, x^2, x^3, x^4 of [14] (in their original indices convention) or with the generators x^{-2}, x^{-1}, x^1, x^2 in the convention of ref. [34] (which has been heavily used by the author).

The algebra and the $\star$-structure are covariant under, i.e. preserved by, matrix multiplication

$$x \to a\,x\,b$$

by the defining matrices a, b of two copies $SU_q(2)$, $SU_q(2)'$ of the special unitary quantum group, or of two copies $\mathbb{H}_q$, $\mathbb{H}'_q$ of the quaternion quantum group, if the entries of a, b commute with each other and with the entries of x. In other words they are covariant under the (mixed left-right) coactions of $SU_q(2)\otimes SU_q(2)' = Spin_q(4)$ and of $\mathbb{H}_q \otimes \mathbb{H}'_q$. This follows from the fact that the twofold coproduct $\Delta^{(2)}(x) = axb$,

$$\Delta^{(2)}(x^{\alpha\alpha'}) = a^{\alpha\beta} b^{\beta'\alpha'} \otimes x^{\beta\beta'}, \qquad\qquad \text{i.e.} \quad x \xrightarrow{\Delta^{(2)}} a\,x\,b, \tag{2.14}$$

is a $\star$-homomorphism, or equivalently both the the left coaction $x \to a\,x$ and the right one $x \to x\,b$ are. In terms of x^i this takes the form

$$\Delta^{(2)}(x^i) = \mathbf{T}^i_j \otimes x^j, \qquad\qquad \mathbf{T}^i_j := B^i_{\alpha\alpha'} a^{\alpha\beta} b^{\beta'\alpha'} B^{-1}{}^{\beta\beta'}_{j}, \tag{2.15}$$

where $B \equiv (B^a_{\alpha\alpha'})$ is the (diagonal and invertible) matrix entering the linear transformation $x^a = B^a_{\alpha\alpha'} x^{\alpha\alpha'}$. Relation $(2.15)_1$ has the same form as the left coaction of ref. [14] of the quantum group $SO_q(4)$ [and of its extension $\widetilde{SO_q(4)} := SO_q(4)\times GL^+(1)$, the quantum group of rotations and scale transformations in 4 dimensions] on $\mathbb{R}^4_q$. This is no formal coincidence: the $\mathbf{T}^i_j$ fulfill the RTT commutation relations and $\star$-conjugation relations

$$\hat{\mathsf{R}}\,\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_1\mathbf{T}_2\hat{\mathsf{R}}, \qquad\qquad \mathbf{T}^{i\,\star}_j = g^{jj'}\mathbf{T}^{i'}_{j'} g_{i'i} \tag{2.16}$$

and in addition $g_{ii'}\mathbf{T}^i_j\mathbf{T}^{i'}_{j'} = g_{jj'}\mathbf{1}$ if the central element $|a||b|$ is 1 (here $\hat{\mathsf{R}}$ and $g_{ab} = B^{-1}{}^{\alpha\alpha'}_{a} B^{-1}{}^{\beta\beta'}_{b} \epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}$ are the braid matrix and the metric matrix of $SO_q(4)$). These are respectively the defining relations of $\widetilde{SO_q(4)}$ and of the compact quantum subgroup $SO_q(4)$ [14]. We have thus an explicit realization of the equivalences

$$SO_q(4) = SU_q(2)\times SU_q(2)'/\mathbb{Z}_2, \qquad\qquad \widetilde{SO_q(4)} = \mathbb{H}_q\times\mathbb{H}'_q/GL(1).$$

The quotient over $\mathbb{Z}_2$ is due to the invariance of $\mathbf{T}^i_j$ under $(a,b)\to(-a,-b)$.

As we shall recall in section 6, the commutation relations are also invariant under the braided group of translations [30, 32] $\mathbb{R}_q^4$, which is the q-deformed version of the group of translations $\mathbb{R}^4$; the role of composition of translations is played by the socalled braided coaddition. They are in fact covariant under the coaction of the full inhomogenous extension $\widetilde{ISO_q}(4)$ [40] of $\widetilde{SO_q}(4)$ (or quantum Euclidean group in 4 dimensions), which includes q-deformed translations together with scale changes and rotations ($\widetilde{ISO_q}(4)$ can be obtained also by "bosonization" of $\mathbb{R}_q^4$ [30]).

Comparison and links with other formulations. A matrix version of the 4-dim quantum Euclidean space (with no interpretation in terms of q-deformed quaternions) was proposed also in [31]. However, the $\star$-relations and the $SO_q(4)$-coaction are different, i.e. cannot be put both in the form (2.1), (2.14), even by a relabelling of the generators.

The slightly extended $\star$-algebra $\mathcal{A}^{\mathrm{ext}}$ obtained by adding as generators the central elements $1/\left(1+\frac{|x|^2}{\rho^2}\right)$, $\rho \in \mathbb{R}^+$, contains the $\star$-algebra of functions on the quantum 4-sphere S_q^4 proposed in [11] (as a 'suspension' of the algebra of a quantum 3-sphere S_q^3). Define

$$\alpha' = \alpha^\star \frac{2\sqrt{2}}{1+2|x|^2} e^{ia}, \qquad \beta' = \gamma^\star \frac{2\sqrt{2}}{1+2|x|^2} e^{ib}, \qquad z = \frac{1-2|x|^2}{1+2|x|^2}, \tag{2.17}$$

where $\alpha, \gamma, \alpha^\star, \gamma^\star$ fulfill (2.1) and $e^{ia}, e^{ib} \in \mathrm{U}(1)$ are possible phase factors. Then α', β', z fulfill the defining relation (1) of the $C^\star$-algebra considered in ref. [11] (where these elements are respectively denoted as α, β, z), in particular

$$\alpha'\alpha'^\star + \beta'\beta'^\star + z^2 = \mathbf{1}, \tag{2.18}$$

and the invertible function $z(|x|)$ spans $[-1, 1[$, i.e. all the spectrum of z except the eigenvalue $z = 1$, as $|x|$ spans all its spectrum $[0, \infty[$. Viceversa, starting from the latter and enlarging it so that it contains the element $(1+z)/2(1-z) =: |x|^2$ then inverting the above formulae one obtains elements $\alpha, \gamma, \alpha^\star\gamma^\star$ fulfilling our defining relations (2.1).

The redefinitions (2.17) have exactly the form of a stereographic projection of $\mathbb{R}^4$ on a sphere S^4 of unit radius (the square radius is $x \cdot x = 2|x|^2$): S^4 is the sphere centered at the origin and $\mathbb{R}^4$ the subspace $z = 0$ immersing both in a $\mathbb{R}^5$ with coordinates defined by $X \equiv (Re(\alpha'), Im(\alpha'), Re(\beta'), Im(\beta'), z)$. In the commutative theory adjoining the missing point $X = (0, 0, 0, 0, 1)$ of S^4 amounts to adding to $\mathbb{R}^4$ the point at infinity, i.e. to compactifying $\mathbb{R}^4$ to S^4. We can thus regard the transition from our algebra to the one considered in ref. [11] as a compactification of $\mathbb{R}_q^4$ into their S_q^4.

3. Other preliminaries

The $SO_q(4)$-covariant differential calculus [6] (d, Ω^*) on $\mathbb{R}_q^4 \sim \mathbb{H}_q$ is obtained imposing covariant homogeneous bilinear commutation relations (3.1) between the x^a and the differentials $\xi^b := dx^b$. Partial derivatives are introduced through the decomposition $d = \xi^a \partial_a = \xi^{\alpha\alpha'} \partial_{\alpha\alpha'}$ of the ($SO_q(4)$-invariant) exterior derivative. All other commutation relations are derived by consistency with nilpotence and the Leibniz rule. Beside (2.3), we have

$$
\begin{aligned}
x^{\alpha\alpha'}\xi^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}\xi^{\gamma\gamma'}x^{\delta\delta'}, && (3.1)\\
\mathcal{P}_{s}{}^{\alpha\beta}_{\gamma\delta}\mathcal{P}_{s}{}^{\alpha'\beta'}_{\gamma'\delta'}\xi^{\gamma\gamma'}\xi^{\delta\delta'} &= 0 = (\xi\epsilon\xi^T)^{\gamma\delta}\epsilon_{\gamma\delta}, && (3.2)\\
\partial_{\alpha\alpha'}\partial_{\beta\beta'} &= \hat{R}^{\delta\gamma}_{\beta\alpha}\hat{R}^{-1}{}^{\delta'\gamma'}_{\beta'\alpha'}\partial_{\gamma\gamma'}\partial_{\delta\delta'}, && (3.3)\\
\partial_{\alpha\alpha'}x^{\beta\beta'} &= \delta^{\beta}_{\alpha}\delta^{\beta'}_{\alpha'} + \hat{R}^{\beta\delta}_{\alpha\gamma}\hat{R}^{\beta'\delta'}_{\alpha'\gamma'}x^{\gamma\gamma'}\partial_{\delta\delta'}, && (3.4)\\
\partial_{\alpha\alpha'}\xi^{\beta\beta'} &= \hat{R}^{-1}{}^{\beta\delta}_{\alpha\gamma}\hat{R}^{-1}{}^{\beta'\delta'}_{\alpha'\gamma'}\xi^{\gamma\gamma'}\partial_{\delta\delta'}. && (3.5)
\end{aligned}
$$

[An alternative $SO_q(4)$-covariant differential calculus $(\hat{d},\hat{\Omega}^*)$ is obtained replacing $\hat{R}$ by $\hat{R}^{-1}$ in (3.1- 3.5)]. The ξ^i transform under $SO_q(4)$ exactly as the x^i, the ∂_i in the contragradient corepresentation. In terms of x^i,∂_j one can build a special unitary element λ such that

$$
\lambda x^i = q^{-1}x^i\lambda, \qquad \lambda\partial^i = q\partial^i\lambda, \qquad \lambda\xi^i = \xi^i\lambda. \tag{3.6}
$$

We introduce the notation $\partial^{\alpha\alpha'} := \epsilon^{\alpha\beta}\epsilon^{\alpha'\beta'}\partial_{\beta\beta'}$, $\partial \equiv (\partial^{\alpha\alpha'})$. The $\partial^{\alpha\alpha'}$ fulfill the same commutation relations (among themselves) as the $x^{\alpha\alpha'}$, and transform in the same way under the $SO_q(4)$ coaction. As a consequence, the Laplacian $\square := g^{hk}\partial_k\partial_h = \partial^{\alpha\alpha'}\partial_{\alpha\alpha'}$ is $SO_q(4)$-invariant and commutes with the $\partial_{\beta\beta'}$, and

$$
\partial\bar{\partial} = \bar{\partial}\partial = I_2|\partial|^2 \equiv I_2\frac{1}{1+q^2}\square. \tag{3.7}
$$

From (3.4), (3.5) it follows

$$
q^2\partial|x|^2 = x + q^4|x|^2\partial, \qquad \partial\frac{1}{|x|^2} = -\frac{q^{-4}x}{|x|^4} + \frac{q^{-2}}{|x|^2}\partial, \qquad |\partial|^2\frac{1}{|x|^2} = \frac{q^{-4}}{|x|^2}|\partial|^2 - \frac{q^{-6}}{|x|^4}x\cdot\partial \tag{3.8}
$$

Since the rhs of the latter formula applied to $\mathbf{1}$ gives zero, $1/|x|^2$ is harmonic, as in the undeformed case.

We denote as $\mathcal{DC}^*$ ("differential calculus algebra") the algebra (over $\mathbb{C}$) generated by $\mathbf{1}, x^i, \xi^i, \partial_i, \lambda^{\pm 1}$; the elements are differential-operator-valued forms. We also denote as $\tilde{\Omega}^*$ the unital subalgebra generated by $\xi^i, x^i, \lambda^{\pm 1}$, as Ω^* (algebra of differential forms) the unital subalgebra generated by ξ^i, x^i, as Ω^*_S the unital subalgebra generated by $T^{\alpha\alpha'}, dT^{\alpha\alpha'}$, as $\bigwedge^*$ (algebra of exterior forms) the unital subalgebra generated by ξ^i.

As usual we introduce in these algebras a grading $\natural \in \mathbb{N}$ given by the degree in ξ^i, and denote as $\mathcal{DC}^p, \Omega^p$, etc., their components with $\natural = p$. Each of these components is a bimodule of dimension $\binom{4}{p}$ w.r.t. to its 0-component. For instance, since by definition $\Omega^0 = \mathcal{A}$, Ω^p is a $\binom{4}{p}$-dimensional $\mathcal{A}$-bimodule; similarly, $\mathcal{DC}^p$ is a $\binom{4}{p}$-dimensional $\mathcal{H}$-bimodule, where $\mathcal{H} := \mathcal{DC}^0$ (the Heisenberg algebra), generated by the $x^i, \partial_i, \lambda^{\pm 1}$. Any $\bigwedge^p$ carries an irreducible corepresentation of $SO_q(4)$. In particular, as $\binom{4}{4} = 1$ all exterior 4-forms are $SO_q(4)$-invariant and proportional to $d^4x := \xi^1\xi^2\xi^3\xi^4$.

All this is exactly as in the case $q=1$, except that as a $C(SU_q(2))$-bimodule Ω^p_S is $\binom{4}{p}$-dimensional when $q \neq 1$ and $\binom{3}{p}$-dimensional when $q=1$.

The whole set of commutation relations (2.3), (3.1- 3.5) is [7] invariant under the replacement $x^{\alpha\alpha'}/|x|^2q^2(1-q^2) \to \partial^{\alpha\alpha'}$. As a corollary, on Ω^* one can realize the action of the exterior derivative as the (graded) commutator

$$d\omega_p = [-\theta, \omega_p\} := -\theta\omega_p + (-)^p\omega_p\theta, \qquad\qquad \omega_p \in \Omega^p \tag{3.9}$$

with the special $SO_q(4)$-invariant 1-form [43] (the 'Dirac Operator', in Connes' [8] parlance)

$$\theta := (d|x|^2)|x|^{-2}\frac{1}{q^2-1} = \frac{q^{-2}}{q^2-1}\xi^{\alpha\alpha'}\frac{x^{\beta\beta'}}{|x|^2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}. \tag{3.10}$$

θ is closed and singular in the $q \to 1$ limit. Applying d to (2.11) we find

$$x\bar{\xi} + \xi\bar{x} = (q^2-1)\theta|x|^2I_2, \qquad\qquad \bar{x}\xi + \bar{\xi}x = (q^2-1)\theta|x|^2I_2. \tag{3.11}$$

Relation (3.1) implies $|x|^2\xi^i = q^2\xi^i|x|^2$, which we supplement with the compatible ones

$$q|x|^{-1}\xi^i = \xi^i|x|^{-1}, \qquad \Rightarrow \qquad q\,|x|^{-1}\,\theta = \theta\,|x|^{-1}. \tag{3.12}$$

By a straightforward computation one also finds

$$dT^{\alpha\alpha'} = q^{-1}\xi^{\alpha\alpha'}\frac{1}{|x|} + (q^{-1}-1)\theta T^{\alpha\alpha'}. \tag{3.13}$$

By (2.14) the 1-form-valued 2×2 matrices $(dT)\overline{T}$, $(d\overline{T})T$ are manifestly invariant under respectively the right and left coaction of $SU_q(2)$, or equivalently the $SU_q(2)'$ and the $SU_q(2)$ part of $SO_q(4)$ coaction. Setting $Q := -\epsilon^{-1}\epsilon^T$ one finds

$$\mathrm{tr}[Q(dT)\overline{T}] = \mathrm{tr}[Q^{-1}(d\overline{T})T] = (q-1)(q-q^{-2})\theta;$$

from (3.10) we see that only in the $q \to 1$ limit these traces vanish. That's why for generic $q \neq 1$ the four matrix elements of either $(dT)\overline{T}$ or $(d\overline{T})T$ are independent, and make up alternative bases for both Ω^*_S and Ω^*.

Actually, one can check (we will give details in [22]) that (d, Ω^*) coincides with the bicovariant differential calculus on $M_q(2), GL_q(2)$ [38, 39], and (d, Ω^*_S) coincides with the Woronowicz 4D- bicovariant one [47, 37] on $C(SU_q(2))$.

One major problem in the present $q \in \mathbb{R}$ case is that the calculus is not real: there is no $\star$-structure such that $d(f^\star) = (df)^\star$, nor is there a $\star$-structure $\star : \Omega^* \to \Omega^*$. Formally, a $\star$-structure would map the commutation relations of (d, Ω^*) into the ones of $(\hat{d}, \hat{\Omega}^*)$, and conversely. At least, there is a $\star$-structure [35]

$$\star : \mathcal{DC}^* \to \mathcal{DC}^*$$

having the desired commutative limit (the $\star$-structure of the De Rham calculus on $\mathbb{R}^4$), but a rather nonlinear character (incidentally, the latter has been recently [19] recast in a much more suggestive form), in other words objects of the second calculus can be realized nonlinearly in terms of objects of the first (and conversely).

The **Hodge map** is a $SO_q(4)$-covariant, $\mathcal{A}$-bilinear map $*:\tilde{\Omega}^p \to \tilde{\Omega}^{4-p}$ [18] such that $*^2 = \mathrm{id}$, defined by

$$^*(\xi^{i_1}\dots\xi^{i_p}) = c_p\,\xi^{i_{p+1}}\dots\xi^{i_4}\varepsilon_{i_4\dots i_{p+1}}{}^{i_1\dots i_p}\lambda^{2p-4},$$

where $\varepsilon^{hijk} \equiv q$-epsilon tensor [18, 17] and c_p are suitable normalization factors. Actually this extends [18] to a $\mathcal{H}$-bilinear map $*:\mathcal{DC}^p \to \mathcal{DC}^{4-p}$ with the same features. For $p=2$ λ-powers disappear and one even gets maps $*:\Omega^2 \to \Omega^2$, $*:\mathcal{DC}^2 \to \mathcal{DC}^2$. The previous equation becomes

$$^*f^{\alpha\beta} = f^{\alpha\beta} \qquad\qquad ^*f'^{\alpha'\beta'} = -f'^{\alpha'\beta'} \tag{3.14}$$

in terms of the "selfdual exterior 2-forms"

$$f^{\alpha\beta} := \mathcal{P}_{s\,\gamma\delta}^{\alpha\beta}\,\epsilon_{\gamma'\delta'}\xi^{\gamma\gamma'}\xi^{\delta\delta'} = \epsilon_{\gamma'\delta'}\xi^{\alpha\gamma'}\xi^{\beta\delta'} = (\xi\epsilon\xi^T)^{\alpha\beta} \tag{3.15}$$

and of the "antiselfdual exterior 2-forms"

$$f'^{\alpha'\beta'} := \mathcal{P}_{s\,\gamma'\delta'}^{\alpha'\beta'}\,\epsilon_{\gamma\delta}\xi^{\gamma\gamma'}\xi^{\delta\delta'} = \epsilon_{\alpha\beta}\xi^{\alpha\alpha'}\xi^{\beta\beta'} = (\xi^T\epsilon\xi)^{\alpha'\beta'}. \tag{3.15$'$}$$

Instead of $f^{\alpha\beta}$ (resp. $f'^{\alpha'\beta'}$) we shall also adopt the matrix elements of $\xi\bar{\xi}$ (resp. $\bar{\xi}\xi$), because

$$(\xi\bar{\xi})^{\alpha\beta} = f^{\alpha\gamma}\epsilon^{\gamma\beta}, \qquad\qquad (\bar{\xi}\xi)^{\alpha'\beta'} = \epsilon^{\alpha'\gamma'}f'^{\gamma'\beta'}. \tag{3.16}$$

As when $q=1$, (only) three out of the four matrix elements $f^{\alpha\beta}$ (resp. $f'^{\alpha'\beta'}$) are independent, because (3.2) implies $\epsilon_{\alpha\beta}f^{\alpha\beta} = 0 = \epsilon_{\alpha'\beta'}f'^{\alpha'\beta'}$. Together, these $f^{\alpha\beta}, f'^{\alpha'\beta'}$ form a basis of the 6-dimensional $\mathcal{A}$-bimodule (resp. $\mathcal{H}$-bimodule) Ω^2 (resp. $\mathcal{DC}^2$). Using relations (3.2) and (2.5) one easily derives the following relations

$$x^{\alpha\alpha'}f^{\beta\gamma} = q(\hat{R}_{12}\hat{R}_{23})^{\alpha\beta\gamma}_{\lambda\mu\nu}f^{\lambda\mu}x^{\nu\alpha'}, \tag{3.17}$$

$$\partial^{\alpha\alpha'}f^{\beta\gamma} = q^{-1}(\hat{R}_{12}\hat{R}_{23})^{\alpha\beta\gamma}_{\lambda\mu\nu}f^{\lambda\mu}\partial^{\nu\alpha'}. \tag{3.18}$$

The second is obtained from the first and Remark 1. In (3.15)$'$ and in the sequel we label a formula regarding antiselfdual 2-forms adding a prime to the label of its selfdual counterpart and omit it, when it can be obtained from the former by the obvious replacements. As another example,

$$x^{\alpha\alpha'}f'^{\beta'\gamma'} = q(\hat{R}_{12}\hat{R}_{23})^{\alpha'\beta'\gamma'}_{\lambda'\mu'\nu'}f'^{\lambda'\mu'}x^{\alpha\nu'}. \tag{3.17$'$}$$

From the previous three formulae and (3.18)$'$ it follows that Ω^2 (resp. $\mathcal{DC}^2$) splits into the direct sum of $\mathcal{A}$- (resp. $\mathcal{H}$-) bimodules

$$\Omega^2 = \check{\Omega}^2 \oplus \check{\Omega}^{2\prime} \qquad\quad (\text{resp. } \mathcal{DC}^2 = \check{\mathcal{DC}}^2 \oplus \check{\mathcal{DC}}^{2\prime}) \tag{3.19}$$

of the eigenspaces of $*$ with eigenvalues $1, -1$ respectively. In [21] we prove that

$$\omega_2\,\omega_2' = \omega_2'\,\omega_2 = 0 \tag{3.20}$$

for any $\omega_2 \in \check{\Omega}^2$, $\omega_2' \in \check{\Omega}^{2\prime}$, (resp. $\omega_2 \in \check{\mathcal{DC}}^2$, $\omega_2' \in \check{\mathcal{DC}}^{2\prime}$)

The 2-forms $(\xi\bar{\xi})^{\alpha\beta}$, $(\bar{\xi}\xi)^{\alpha'\beta'}$ are exact. 1-form-valued matrices a, a' such that

$$d\,a = \xi\bar{\xi}, \qquad\qquad d\,a' = \bar{\xi}\xi \tag{3.21}$$

are clearly defined up to d-exact terms. One can choose

$$a_\kappa := -\xi\bar{x} + \kappa\,\theta|x|^2\,I_2 \tag{3.22}$$

with complex κ. If $\kappa \neq \kappa_0 := q^2(q^2-1)/(q^2+1)$ the four matrix elements of a_κ are all independent and make up an alternative basis for Ω^1; they belong to the $(3,1)\oplus(1,1)$-dimensional (reducible) corepresentation of $SU_q(2)\times SU_q'(2)$. (And similarly for $\hat{a}'_\kappa$). Whereas there are only three independent

$$a_{\kappa_0}{}^{\alpha\beta} = \mathcal{P}_s{}^{\alpha\lambda}_{\gamma\delta}(\xi\epsilon x^T)^{\gamma\delta}\epsilon^{\beta\delta}, \tag{3.23}$$

because $a_{\kappa_0}{}^{\alpha\beta}(\epsilon\epsilon^T)_{\beta\alpha} = 0$; the latter belong to the (3,1) irreducible corepresentation of $SU_q(2)\times SU_q'(2)$. In the $q=1$ limit (3.23) becomes the familiar

$$a_{\kappa_0}{}^{\alpha\beta} = -\Big(\xi\epsilon^{-1}x^T\Big)^{(\alpha\lambda)}\epsilon^{\lambda\beta} = -\left\{Im(\xi\,\bar{x})\right\}^{\alpha\beta},$$

where $(\alpha\lambda)$ denotes symmetrization w.r.t. α,λ, and Im the imaginary part. Another peculiar choice is $\kappa_+ = q-1$, which gives $a_{\kappa_+} = -q(dT)\overline{T}|x|^2$, whence the simple change

$$\overline{T}[(dT)\overline{T}]T = \overline{T}(dT) \tag{3.24}$$

under the 'similarity' transformation T, as when $q=1$. Since $(\bar{\xi}\xi)^{\alpha'\beta'}$ belongs to $\check{\Omega}^{2\prime}$, which is a $\mathcal{A}$-bimodule, we also find [using (3.12) and (3.11)]

$$T\bar{\xi}\xi\overline{T} = \xi\bar{\xi}q^2 + (q^{-2}-q^2)\xi\bar{x}\theta\ \in \check{\Omega}^{2\prime}. \tag{3.25}$$

4. Formulations of NC gauge theories

We recall some minimal common elements in the formulations of $\mathrm{U}(n)$ gauge theories on commutative as well as noncommutative spaces [8, 29] (see also [27, 16]). We denote by $\mathcal{A}$ the '$\star$-algebra of functions on the noncommutative space' under consideration, by (d,Ω^*) a differential calculus on $\mathcal{A}$, real in the sense that $d(f^\star) = (df)^\star$. In $\mathrm{U}(n)$ gauge theory the gauge transformations U are unitary $\mathcal{A}$-valued $n\times n$ matrices, $U\in M_n(\mathcal{A})\equiv M_n(\mathbb{C})\otimes_{\mathbb{C}}\mathcal{A}$ with $U^\dagger = U^{-1}$. The gauge potential $A\equiv(A^{\dot\alpha}_{\dot\beta})$ is an antihermitean 1-form-valued $n\times n$ matrix, $A\in M_n(\Omega^1(\mathcal{A}))$ with $A^\dagger = -A$. The associated field strength $F\in M_n(\Omega^2)$ and covariant derivative $D: M_n(\Omega^p)\to M_n(\Omega^{p+1})$ are defined as usual by

$$F := dA + AA \qquad\qquad D\omega_p := d\omega_p + [A,\omega_p\}, \tag{4.1}$$

and are therefore hermitean, in the sense $F^\dagger = F$, $D(f^\dagger) = (Df)^\dagger$. At the right-hand side the product AA has to be understood both as a (row by column) matrix product and as a wedge product. Even for $n=1$ (electromagnetism) $AA\neq 0$, contrary to the

JHEP02(2007)010

commutative case. The Bianchi identity $DF = dF + [A, F] = 0$ is automatically satisfied and the Yang-Mills equation reads as usual $D^*F = 0$. Because of the Bianchi identity, in a 4D Riemannian geometry endowed with a Hodge map $*$ the latter is automatically satisfied by any solution of the (anti)self-duality equations

$$^*F = \pm F. \tag{4.2}$$

If Ω^2 splits as in (3.19) then F is uniquely decomposed in a selfdual and an antiselfdual part,

$$F = F^+ + F^-. \tag{4.3}$$

The Bianchi identity, the Yang-Mills equation, the (anti)self-duality equations, the flatness condition $F = 0$ are preserved by gauge transformations

$$A^U = U^{-1}(AU + dU), \qquad \Rightarrow \qquad F^U = U^{-1}FU. \tag{4.4}$$

As usual, $A = U^{-1}dU$ implies $F = 0$. If the exterior derivative can be realized as the graded commutator (3.9) with a special 1-form [8, 47, 29] $-\theta$, then introducing the 1-form-valued matrix $B := -\theta I_n + A$ one finds that

$$F = BB, \qquad D = [B, \cdot\,\} \tag{4.5}$$

and Bianchi identity is now even more trivial. In Connes' noncommutative geometry $-\theta$ is the 'Dirac operator', which has to fulfill more stringent requirements [8].

Up to normalization factors, the gauge invariant 'action' S and 'Pontryagin index' (or 'second Chern number') $\mathcal{Q}$ are defined by

$$S = \mathrm{Tr}(F\,{}^*F), \qquad \mathcal{Q} = \mathrm{Tr}(FF) \tag{4.6}$$

where Tr stands for a positive-definite trace combining the $n \times n$-matrix trace with the integral over the noncommutative manifold (as such, Tr has to fulfill the cyclic property). If integration $\int$ fulfills itself the cyclic property then this is obtained by simply choosing $\mathrm{Tr} = \int \mathrm{tr}$, where tr stands for the ordinary matrix trace. If, as in the case under discussion, (3.20) holds, $S, \mathcal{Q}$ respectively split into the sum, difference of two nonnegative contributions:

$$S = \mathrm{Tr}(F^{+*}F^+) + \mathrm{Tr}(F^{-*}F^-), \qquad \mathcal{Q} = \mathrm{Tr}(F^{+}\,{}^*F^+) - \mathrm{Tr}(F^{-}\,{}^*F^-). \tag{4.7}$$

As in the commutative case, these relations imply $S \geq |\mathcal{Q}| \geq 0$.

In commutative geometry the socalled Serre-Swan theorem [42, 9] states that vector bundles over a compact manifold coincide with finitely generated projective modules $\mathcal{E}$ over $\mathcal{A}$. The gauge connection A of a gauge group (fiber bundle) acting on a vector bundle is expressed in terms of the projector $\mathcal{P}$ characterizing the projective module. Therefore these projectors can be used to completely determine the connections. In Connes' standard approach [8] to noncommutative geometry the finitely generated projective modules are the primary objects to define and develop the gauge theory. The topological properties of the connections can be classified in terms of topological invariants (Chern numbers), and the latter can be computed directly in terms of characters of $\mathcal{P}$ (Chern-Connes characters), in particular $\mathcal{Q}$ can be computed in terms of the second Chern-Connes character, when Connes' formulation of noncommutative geometry applies.

In the present $\mathcal{A} \equiv C(\mathbb{R}^4_q) = C(\mathbb{H}_q)$ case there are **2 main problems** preventing the application of this formulation of gauge theories:

i) Integration over $\mathbb{R}^4_q$ fulfills a *deformed* cyclic property [43].

ii) $d(f^\star) \neq (df)^\star$, and there is no $\star$-structure $\star : \Omega^* \to \Omega^*$, but (as mentioned in section 3) only a $\star$-structure $\star : \mathcal{DC}^* \to \mathcal{DC}^*$ [35], with a nonlinear character.

On the basis of our results [19] we hope that both problems could be solved

i) allowing for $\mathcal{DC}^1$-valued A ($\Rightarrow \mathcal{DC}^2$-valued F's), and/or

ii) realizing $\mathrm{Tr}(\cdot)$ by in the form $\mathrm{Tr}(\cdot){:=}\!\int \mathrm{tr}(W\cdot)$, with W some suitable positive definite $\mathcal{H}$-valued (i.e. pseudo-differential-operator-valued) $n\times n$ matrix (this implies a change in the hermitean conjugation of differential operators), or even a more general form.

5. q-deformed SU(2) instanton

We look for $A \in M_2(\Omega^1)$ solutions of the (anti)self-duality equations (4.2) virtually yielding a finite action functional (4.6). Among them we expect deformations of the (multi)instanton solutions of $su(2)$ Yang-Mills theory on the "commutative" $\mathbb{R}^4$. We first recall the instanton solution of Belavin *et al.* [4], which we write down both in t' Hooft [44] and in ADHM [2] quaternion notation:

$$\begin{aligned} A &= dx^i\,\sigma^a\,\underbrace{\eta^a_{ij}x^j\frac{1}{\rho^2+r^2/2}}_{A^a_i} = -Im\left\{\xi\,\frac{\bar{x}}{|x|^2}\right\}\frac{1}{1+\rho^2\frac{1}{|x|^2}} \\ &= -(dT)\overline{T}\frac{1}{1+\rho^2\frac{1}{|x|^2}}, \\ F &= \xi\bar{\xi}\,\rho^2\frac{1}{(|x|^2+\rho^2)^2}. \end{aligned} \tag{5.1}$$

Here $r^2 := x\cdot x = 2|x|^2$, σ^a are the Pauli matrices, η^a_{ij} are the so-called t' Hooft η-symbols, ρ is the size of the instanton (here centered at the origin). The third equality is based on the identity

$$\xi\,\frac{\bar{x}}{|x|^2} = (dT)\overline{T} + I_2\frac{d|x|^2}{2|x|^2}$$

and the observation that the first and second term at the rhs are respectively antihermitean and hermitean, i.e. the imaginary and the real part of the quaternion.

In terms of the modified gauge potential $B := A{-}\theta I_2$ a natural Ansatz for the deformed instanton solution in the 'regular gauge' is (in matrix notation)

$$B := A - \theta I_2 = \xi\frac{\bar{x}}{|x|^2}\,l + \theta\,I_2\,m, \tag{5.2}$$

where l, m are functions of x only through $|x|$. For any $f(x)$ we shall denote $f_q(x) := f(qx)$. Using $\theta^2 = 0$ and (3.12), (3.11), (3.9), (2.11) we find

$$F = B^2 = \xi\bar{\xi}(m-l)\, l_q \frac{q^{-2}}{|x|^2} + \xi\theta\bar{x}\left[(q^2-1)l_q l + l_q m - q^2 m_q l\right]\frac{q^{-2}}{|x|^2}.$$

A sufficient condition for F to be selfdual is that the expression in the square bracket vanishes. Setting $h := m/l$ this amounts to the equation $q^2 h_q - h = (q^2-1)$, which is solved by $m = \left[1 + \bar{\rho}^2/|x|^2\right] l$, where $\bar{\rho}^2$ is a constant, or might be a further generator of the algebra, commuting with θ. Replacing in the expression for A, F, we find a family of solutions

$$A_l = q(dT)\overline{T}l + \theta I_2\left\{1 + \left[q + \bar{\rho}^2\tfrac{1}{|x|^2}\right] l\right\}, \qquad F_l = \xi\bar{\xi}\tfrac{1}{|x|^2}\bar{\rho}^2\tfrac{q^{-2}}{|x|^2} l_q l, \tag{5.3}$$

parametrized by the function $l(|x|)$. This large (compared to the undeformed case) freedom in the choice of the solution is due to the fact that we have not yet imposed on A the antihermiticity condition. Actually, we don't know yet what the 'right' antihermiticity condition is: in fact, for no l is A antihermitean w.r.t. the $\star$-structure [35] mentioned in section 3. In any case, one should check that for the final A the resulting F decreases faster than $|x|^{-2}$ at infinity, so that the resulting action functional (4.6) is finite.

The second term in $(5.3)_1$ is proportional to $d|x|^2$; in the commutative limit $q = 1$ it is a connection associated to the noncompact factor $GL^+(1)$ of $\mathbb{H}$. In this limit the antihermiticity condition on A amounts to the vanishing of this term and completely determines the solution. It factors $GL^+(1)$ out of the gauge group to leave a pure SU(2) gauge theory. In the q-deformed case, as we still ignore what the 'right' $\star$- (i.e. Hermitean) structure could be, it could well happen that w.r.t. the latter the second term in $(5.3)_1$ contains also a antihermitean (i.e. imaginary) part, which would be the connection associated to an additional U(1) factor of the gauge group and which could not be consistently disposed of. In the latter case the associated gauge theory would necessarily be a deformed U(2) one.

For the moment we cannot solve the ambiguity, and content ourselves with writing the solution for a couple of selected choices of l. If we choose l so that the second term in $(5.3)_1$ vanishes and set $\rho^2 = \bar{\rho}^2 q^{-1}$ we obtain

$$A = -(dT)\overline{T}\frac{1}{1+\rho^2\frac{1}{|x|^2}} \qquad F = q^{-1}\xi\bar{\xi}\frac{1}{q^2|x|^2+\rho^2}\rho^2\frac{1}{|x|^2+\rho^2}. \tag{5.4}$$

This has manifestly the desired $q \to 1$ limit (5.1). The second choice,

$$l = -\frac{1+q^2}{1+q^4}\frac{1}{1+\tilde{\rho}^2\frac{1}{|x|^2}} \qquad \tilde{\rho}^2 := \frac{1+q^2}{1+q^4}\bar{\rho}^2,$$

is designed in order that A is proportional to the $a_{\kappa 0}$ of (3.23), so that $A^{\alpha\beta}$ span the (3,1) dimensional, irreducible corepresentation of $SU_q(2) \times SU'_q(2)$. The result is:

$$\tilde{A} = -\frac{1+q^2}{1+q^4}a_{\kappa 0}\frac{1}{|x|^2+\tilde{\rho}^2} \qquad \tilde{F} = \frac{1+q^2}{1+q^4}\xi\bar{\xi}\frac{1}{q^2|x|^2+\tilde{\rho}^2}\tilde{\rho}^2\frac{1}{|x|^2+\tilde{\rho}^2}. \tag{5.5}$$

This also has the desired $q \to 1$ limit (5.1). If $\bar{\rho}^2 \neq 0$, in both cases FF is regular everywhere and decreases as $1/|x|^8$ as $x \to \infty$, therefore it virtually will yield finite action S and Pontryagin index $\mathcal{Q}$ upon integration.

JHEP02(2007)010

As in the undeformed case, to make the determination of multi-instanton solutions easier it is useful to go to the "singular gauge". Note that as in the $q=1$ case $T=x/|x|$ is unitary and formally not continuous at $x=0$, so it can play the role of a 'singular gauge transformation'. In fact A can be obtained through the gauge transformation $A = T(\hat{A}\overline{T}+d\overline{T})$ from the "singular" gauge potential

$$\hat{A} = \overline{T}dT\frac{1}{1+|x|^2\frac{1}{\rho^2}} \tag{5.6}$$

$$\stackrel{(3.13)}{=} -\left[q^{-1}\bar{\xi}\frac{x}{|x|^2}-\frac{q^{-3}}{q+1}\xi^{\alpha\alpha'}\frac{x^{\beta\beta'}}{|x|^2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}\right]\frac{1}{1+|x|^2\frac{1}{\rho^2}} \tag{5.7}$$

$$\hat{F} = \overline{T}q^{-1}\xi\bar{\xi}\frac{1}{q^2|x|^2+\rho^2}\rho^2\frac{1}{|x|^2+\rho^2}T, \tag{5.8}$$

which is the analog of the instanton solution in the "singular gauge" found by 't Hooft in [44]. $\hat{A}$ is singular in that it has a pole in $|x|=0$. More generally, the generic solution (5.3) can be obtained through the gauge transformation $A_l = T(\hat{A}_l\overline{T}+d\overline{T})$ from a singular solution $\hat{A}_l$. The latter can be obtained also by starting from an Ansatz like $\hat{B}=\bar{\xi}\frac{x}{|x|^2}\,\hat{l}+\theta\, I_2\,\hat{m}$, instead of (5.2), and imposing that the $\bar{\xi}\xi$ and the $\bar{\xi}\theta x$ term in $\hat{F}=\hat{B}^2$ appear in a combination proportional to (3.25).

A straightforward computation by means of (3.8) shows that $\hat{A}$ can be expressed also in the form

$$\hat{A} = (\hat{\mathcal{D}}\phi)\phi^{-1}, \tag{5.9}$$

where $\hat{\mathcal{D}}$ is the first-order-differential-operator-valued 2×2 matrix obtained from the expression in the square bracket in (5.7) by the replacement $x^{\alpha\alpha'}/|x|^2 \to q^4\partial^{\alpha\alpha'}$,

$$\hat{\mathcal{D}} := q^3\bar{\xi}\partial - \frac{q}{q+1}dI_2, \tag{5.10}$$

(for simplicity we are here assuming that ρ^2 commutes with $\xi\partial$) and ϕ is the harmonic potential

$$\phi := 1+\rho^2\frac{1}{|x|^2}, \qquad \Box\phi = 0.$$

This is the analog of what happens in the classical case.

The **anti-instanton solution** is obtained just by converting unbarred into barred matrices, and conversely, as in the $q=1$ case. For instance, from (5.4) we obtain the anti-instanton solution in the regular gauge

$$A' = -(d\overline{T})T\,\frac{1}{1+\rho^2\frac{1}{|x|^2}}, \qquad F' = q^{-1}\bar{\xi}\xi\frac{1}{|x|^2+\rho^2}\rho^2\frac{1}{q^2|x|^2+\rho^2}, \tag{5.11}$$

and for the one in the singular gauge $\hat{A}' = (\hat{\mathcal{D}}'\phi)\phi^{-1}$, where

$$\hat{\mathcal{D}}' := q^3\xi\bar{\partial} - \frac{q}{q+1}dI_2. \tag{5.12}$$

Recovering the instanton projective module of [11]. In commutative geometry the instanton projective module $\mathcal{E}$ over $\mathcal{A}$ and the associated gauge connection can be most easily obtained using the quaternion formalism, in the way described e.g. in ref. [1]. $\mathbb{H} \sim \mathbb{R}^4$ can be compactified as $P^1(\mathbb{H}) \sim S^4$. Let $(w, x) \in \mathbb{H}^2$ be homogenous coordinates of the latter, and choose $w = I_2$ on the chart $\mathbb{H} \sim \mathbb{R}^4$. The element $u \in \mathbb{H}^2$ defined by

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_2 \\ \frac{\rho x}{|x|^2} \end{pmatrix} \left(1+\frac{\rho^2}{|x|^2}\right)^{-1/2} \tag{5.13}$$

fulfills $u^\dagger u = I_2\mathbf{1}$, and the 4×2 $\mathcal{A}$-valued matrix u has only three independent components. Therefore the 4×4 $\mathcal{A}$-valued matrix

$$\mathcal{P} := uu^\dagger = \begin{pmatrix} I_2 & \frac{\rho\bar{x}}{|x|^2} \\ \frac{\rho x}{|x|^2} & \frac{\rho^2}{|x|^2}I_2 \end{pmatrix} \frac{1}{1+\frac{\rho^2}{|x|^2}} \tag{5.14}$$

is a self-adjoint three-dimensional projector. It is the projector associated in the Serre-Swan theorem correspondence to the gauge connection (5.6), by the formula $\hat{A} = u^\dagger du$. The associated projective module $\mathcal{E}$ is embedded in the free module $\mathcal{A}^{16}$ seen as $M_4(\mathcal{A})$, and is obtained from the latter as $\mathcal{E} = \mathcal{P}M_4(\mathcal{A})$.

In the present q-deformed setting we immediately check that the element $u \in \mathbb{H}_q^2$ defined by (5.13) fulfills $u^\dagger u = I_2\mathbf{1}$ again, so that the 4×2 $\mathcal{A}$-valued matrix $\mathcal{P}$ defined by (5.14) is hermitean and idempotent, and has only 3 independent components. Therefore, it defines the 'instanton projective module' $\mathcal{E} = \mathcal{P}M_4(\mathcal{A})$ also in the q-deformed case. One can easily verify that $\mathcal{P}$ reduces to the hermitean idempotent e of [11] if one chooses the instanton size as $\rho = 1/\sqrt{2}$ and performs the change of generators (2.17). Therefore, interpreting the model [11] as a compactification to S_q^4 of ours, we can use all the results [11] about the Chern-Connes classes of e.

Unfortunately in the q-deformed case it is no more true that $\hat{A} = u^\dagger du$, essentially because the $|x|$-dependent global factor multiplying the matrix at the rhs(5.14) does not commute with the 1-forms of the present calculus ($|x|\xi^i = q\xi^i|x|$).

6. Changing size and center of the (anti)instanton

Applying the $\widetilde{SO_q(4)}$ coaction (2.14) to the instanton gauge potentials (5.3) we find

$$A_l(\xi, x) \xrightarrow{\Delta^{(2)}} aA_l(\xi|c|, x|c|)a^{-1}, \qquad\qquad F_l(\xi, x) \xrightarrow{\Delta^{(2)}} aF_l(\xi|c|, x|c|)a^{-1}. \tag{6.1}$$

where $|c|^2 := |a|^2|b|^2$. The result is the same also if we consider $|c|^2$ as an independent parameter and choose a, b with $|a|=|b|=\mathbf{1}$. In particular, on (5.4)

$$A \xrightarrow{\Delta^{(2)}} -a\,(dT)\overline{T}\frac{1}{1+\rho'^2\frac{1}{|x|^2}}a^{-1}, \qquad\qquad F \xrightarrow{\Delta^{(2)}} a\,\xi\bar{\xi}\frac{q^{-1}\rho'^2}{q^2|x|^2+\rho'^2}\frac{1}{|x|^2+\rho'^2}a^{-1}, \tag{6.2}$$

where we have set $\rho'^2 := \rho^2|c|^{-2}$. These gauge potentials are again solutions of the self-duality equation, since the latter is covariant under the $\widetilde{SO_q(4)}$ coaction. The result of the $SO_q(4)$ coaction ($|a| = |b| = \mathbf{1}$) can be reabsorbed into a (global) gauge transformation (4.4), with $U = a$ (and similarly $U = \bar{b}$ for the anti-instanton gauge potentials), i.e. is a gauge equivalent solution. Note that we are thus introducing gauge transformations depending on the additional noncommuting parameters a, b. A full $\widetilde{SO_q(4)}$ coaction ($|c| \neq 1$) instead involves also a change of the size of the instanton, and gives an inequivalent solution. We can thus obtain any size starting from the instanton with unit size.

Having built an (anti)instanton "centered at the origin" with arbitrary size one would like first to translate the latter in space to another point y, then to construct n-instanton solutions "centered at points y_μ", $\mu = 1, 2, \ldots, n$. The appropriate framework is to replace tensor products $\otimes$ by braided tensor products $\underline{\otimes}$ and apply the braided coaddition [32] to the covectors x. This gives new (i.e. gauge inequivalent) solutions. The braided coaddition [32] of the coordinates x reads $\underline{\Delta}(x) = x\underline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}x \equiv x - y$, where we have renamed $x := x\underline{\otimes}\mathbf{1}$, $y := -\mathbf{1}\underline{\otimes}x$. It follows $y\bar{y} = \bar{y}y = I_2|y|^2$. Out of the two possible braidings we choose the following one:

$$
\begin{aligned}
y^{\alpha\alpha'}x^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}x^{\gamma\gamma'}y^{\delta\delta'}, \\
\partial_{\alpha\alpha'}y^{\beta\beta'} &= \hat{R}^{\beta\delta}_{\alpha\gamma}\hat{R}^{\beta'\delta'}_{\alpha'\gamma'}y^{\gamma\gamma'}\partial_{\delta\delta'}, \\
y^{\alpha\alpha'}\xi^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta}\hat{R}^{\alpha'\beta'}_{\gamma'\delta'}\xi^{\gamma\gamma'}y^{\delta\delta'}.
\end{aligned}
\tag{6.3}
$$

We also enlarge the algebra by introducing further generators $1/|y|$, $1/|z|$ (where $z := x - y$) fulfilling relations

$$
\begin{aligned}
&q\gamma\xi^i = \xi^i\gamma, && \text{for} \quad \gamma = \tfrac{1}{|x|}, \tfrac{1}{|y|}, \tfrac{1}{|z|} \\
&y^i\tfrac{1}{|x|} = \tfrac{q}{|x|}y^i, && x^i\tfrac{q}{|y|} = \tfrac{1}{|y|}x^i, \\
&\tfrac{1}{|y|}\tfrac{1}{|x|} = \tfrac{q}{|x|}\tfrac{1}{|y|}, && \\
&\tfrac{1}{|z|}\tfrac{x^i}{|x|^2} = \tfrac{x^i}{|x|^2}\tfrac{q}{|z|} + (1-q)\tfrac{z^i}{|z|^3}, && \tfrac{q}{|z|}\tfrac{y^i}{|y|^2} = \tfrac{y^i}{|y|^2}\tfrac{1}{|z|} + (1-q)\tfrac{z^i}{|z|^3}.
\end{aligned}
\tag{6.4}
$$

These are (the only) consistent extensions of the previous relations to the inverse square root of $|z|^2, |x|^2, |y|^2$ having the desired, commutative $q \to 1$ limit.

Under the replacement $x \to x - y$ (i.e. under $\underline{\Delta}$) the differential calculus is invariant, implying that solutions are mapped into solutions. Therefore the instanton solution with "shifted" center y will read in the regular gauge

$$
A = -(dT)\overline{T}\frac{1}{1 + \rho^2\frac{1}{|x-y|^2}} \qquad\qquad F = q^{-1}\xi\bar{\xi}\frac{1}{q^2|x-y|^2 + \rho^2}\rho^2\frac{1}{|x-y|^2 + \rho^2}. \tag{6.5}
$$

and in the singular gauge

$$
\begin{aligned}
\hat{A} &= (\hat{\mathcal{D}}\phi)\phi^{-1}, \\
\phi &:= 1 + \rho^2\frac{1}{|x-y|^2}, \\
\hat{F} &= \overline{T}q^{-1}\xi\bar{\xi}T\frac{1}{q^2|x-y|^2 + \rho^2}\rho^2\frac{1}{|x-y|^2 + \rho^2}.
\end{aligned}
\tag{6.6}
$$

7. Multi-instanton solutions

On the basis of the latter and of the $q = 1$ results [44, 48], we first look for n-instanton solutions of the self-duality equation in the "singular gauge" in the form (5.9). Beside the coordinates $x^i \equiv -y^i_0$ we introduce n other coordinates y^i_μ, $\mu = 1, 2, \ldots, n$ generating as many $\mathbb{R}^4_q$ and braided to each other:

$$\begin{aligned} y_\mu \bar{y}_\mu &= \bar{y}_\mu y_\mu = I_2 |y_\mu|^2 \\ y^{\alpha\alpha'}_\nu y^{\beta\beta'}_\mu &= \hat{R}^{\alpha\beta}_{\gamma\delta} \hat{R}^{\alpha'\beta'}_{\gamma'\delta'} y^{\gamma\gamma'}_\mu y^{\delta\delta'}_\nu \end{aligned} \tag{7.1}$$

with $\mu < \nu$ and no sum over repeated μ. We shall call $\mathcal{A}_n$ the larger algebra generated by the y^i_μ's and by parameters ρ_μ, $\mu = 1, \ldots, n$ fulfilling the commutation relations

$$\begin{aligned} \rho^2_\nu \rho^2_\mu &= q^2\, \rho^2_\mu \rho^2_\nu, \qquad \nu < \mu, \\ \rho^2_\nu y^i_\mu &= y^i_\mu \rho^2_\nu \begin{cases} q^{-2} & \nu < \mu, \\ 1, & \nu \geq \mu. \end{cases} \end{aligned} \tag{7.2}$$

We shall also enlarge $\mathcal{A}_n$ to the extended Heisenberg algebra $\mathcal{H}_n$ and extended algebra of differential forms $\Omega^*(\mathcal{A}_n)$ by adding as generators the ∂_i and the ξ^i respectively, and to the extended differential calculus algebra $\mathcal{DC}(\mathcal{A}_n)$ by adding as generators both the ξ^i, ∂_i, with cross commutation relations

$$\begin{aligned} \rho^2_\mu \xi^{\alpha\alpha'} &= \xi^{\alpha\alpha'} \rho^2_\mu, & \partial_{\alpha\alpha'} \rho^2_\mu &= \rho^2_\mu \partial_{\alpha\alpha'}, \\ y^{\alpha\alpha'}_\mu \xi^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta} \hat{R}^{\alpha'\beta'}_{\gamma'\delta'} \xi^{\gamma\gamma'} y^{\delta\delta'}_\mu, & \partial_{\alpha\alpha'} y^{\beta\beta'}_\mu &= \hat{R}^{\beta\delta}_{\alpha\gamma} \hat{R}^{\beta'\delta'}_{\alpha'\gamma'} y^{\gamma\gamma'}_\mu \partial_{\delta\delta'}, \end{aligned} \tag{7.3}$$

Note that the first relations, together with the decomposition $d = \xi^i \partial_i$, imply

$$d\, \rho^2_\mu = \rho^2_\mu d. \tag{7.4}$$

Also, from these relations it is evident that $\check{\Omega}^2(\mathcal{A}_n), \check{\Omega}^{2\prime}(\mathcal{A}_n)$ are $\mathcal{A}_n$-bimodules (resp. $\check{\mathcal{DC}}^2(\mathcal{A}_n), \check{\mathcal{DC}}^{2\prime}(\mathcal{A}_n)$ are $\mathcal{H}_n$-bimodules). Let us introduce the short-hand notation

$$z^{\alpha\alpha'}_\mu := x^{\alpha\alpha'} - v^{\alpha\alpha'}_\mu, \qquad v^{\alpha\alpha'}_\mu := \sum_{\nu=1}^{\mu} y^{\alpha\alpha'}_\nu, \qquad \mu = 1, 2, \ldots, n;$$

$v^{\alpha\alpha'}_\mu$ will play the role of coordinates of the center of the μ-th instanton. It is easy to check from (7.1) that these new n sets of variables generate as many copies of the quantum Euclidean space $\mathbb{R}^4_q$, namely

$$z_\mu \bar{z}_\mu = \bar{z}_\mu z_\mu = |z_\mu|^2 I_2 \tag{7.5}$$

(no sum over repeated μ) and together with $x^{\alpha\alpha'}$ make up an alternative Poincaré-Birkhoff-Witt basis of the algebra $\mathcal{A}_n$, (i.e. ordered monomials in these variables make up a basis of the vector space underlying $\mathcal{A}_n$). Moreover, differentiating $z^{\alpha\alpha'}_\mu$ and commuting it with $\xi^{\beta\beta'}$ is like differentiating and commuting $x^{\alpha\alpha'}$:

$$\begin{aligned} \partial_{\alpha\alpha'} z^{\beta\beta'}_\mu &= \delta^\beta_\alpha \delta^{\beta'}_{\alpha'} + \hat{R}^{\beta\delta}_{\alpha\gamma} \hat{R}^{\beta'\delta'}_{\alpha'\gamma'} z^{\gamma\gamma'}_\mu \partial_{\delta\delta'}, \\ z^{\alpha\alpha'}_\mu \xi^{\beta\beta'} &= \hat{R}^{\alpha\beta}_{\gamma\delta} \hat{R}^{\alpha'\beta'}_{\gamma'\delta'} \xi^{\gamma\gamma'} z^{\delta\delta'}_\mu. \end{aligned}$$

JHEP02(2007)010

Therefore for any $\mu = 1, 2, \ldots, n$ the replacement $x \to z_\mu$ in any true relation involving x, ∂, ξ will generate a new true relation, which we shall label by adding the subscript μ to the original one.

The solution ϕ searched for (5.9) is of the form

$$\phi \equiv \phi_n = 1 + \sum_{\mu=1}^{n} \rho_\mu^2 \frac{1}{|z_\mu|^2}, \tag{7.6}$$

namely a scalar "function" of the coordinates x^i, of the instanton "sizes" ρ_μ and of the "coordinates of their centers". For this to be allowed we have further enlarged $\mathcal{A}_n, \Omega^*(\mathcal{A}_n), \mathcal{H}_n, \mathcal{DC}(\mathcal{A}_n)$ to extended algebras $\mathcal{A}_n^{\text{ext}}, \Omega^*(\mathcal{A}_n^{\text{ext}})\mathcal{H}_n^{\text{ext}}, \mathcal{DC}(\mathcal{A}_n^{\text{ext}})$ by adding as generators inverse square roots $1/|z_\mu|$, but we also add the inverses $1/\phi_m$, together with corresponding commutation relations (see [21]) consistent with the ones given so far. The basic ones can be obtained from the relations of section 6 by the replacements $x \to z_\mu$, $\rho \to \rho_\mu$, $y \to \sum_{\lambda=\mu+1}^{\nu} y_\lambda$, $z \to z_\nu$, $\rho_z \to \rho_\nu$ with $\nu > \mu$. By relations (3.8), $(3.4)_\mu$ ϕ is harmonic, exactly as in the classical case. In Theorem 1 of [21] we prove that $\hat{A} = (\hat{\mathcal{D}}\phi)\phi^{-1}$ fulfills the selfduality equation $(4.2)_1$. Explicitly, the field strength is

$$\hat{F} = \frac{-q^5}{4_q} \left[\epsilon^{-1}(\xi\bar{\xi}\partial)^T \epsilon \partial\phi\right] \left[q\phi^{-1} + \phi_q^{-1}\right] + q^2 \epsilon^{-1}(\xi\bar{\xi}\partial\phi)^T \epsilon(\partial\phi)\phi^{-1}\phi_q^{-1}, \tag{7.7}$$

where $\phi_q(\{z_i\}) := \phi(\{qz_i\})$. (This is a selfdual matrix because $\xi\bar{\xi}$ is.)

Formally, as $x \to \infty$ also $z_\mu \to \infty$, $\phi \to 1$, and a simple inspection shows that $\hat{A} \to 0$ as $1/|x|^3$, $\hat{F} \to 0$ as $1/|x|^4$, exactly as in the case $q = 1$. Therefore $\hat{F}\hat{F}$ decreases fast enough at infinity for integrals like $\int \operatorname{tr}(\hat{F}\hat{F})$ to converge.

On the other hand, as $z_\mu \to 0$ the function ϕ and therefore the gauge potential $\hat{A}$ are singular, i.e. formally diverge. We don't know yet whether the singularity will cause problems also in a proper functional-analytical treatment (this requires analyzing representations of the algebra). If this is the case then, as in the undeformed theory, the question arises if this singularity is only due to the choice of a singular gauge and can be removed by performing a suitable gauge transformation, or it really affects the field strength. We address this issue semi-heuristically. We shall say that an element of our algebra is: 1. analytic in z_μ if its power expansion has no poles in z_μ, i.e. does not depend on $1/|z_\mu|$; regular in z_μ if it formally keeps finite as $z_\mu \to 0$, i.e. in its power expansion the dependence on $1/|z_\mu|$ occurs only through $z_\mu/|z_\mu|$. Since such dependences might change upon changing the order in which the variables $z_1, z_2, \ldots, z_n$, and possible extra variables $1/|z_1 - z_2|, 1/|z_1 - z_3|, \ldots$ (if necessary), are displayed, these conditions have to be met for any order. In the appendix of [21] we show that performing the "singular gauge transformation" U_2 defined by

$$U_2 \equiv U_2(z_1, z_2) := \frac{\bar{z}_1}{|z_1|} \frac{y_2}{|y_2|} \frac{\bar{z}_2}{|z_2|} \tag{7.8}$$

on $\hat{A}_2$ we obtain a 2-istanton solution

$$A_2 = U_2^{-1}\left(\hat{A}U_2 + dU_2\right) \tag{7.9}$$

analytic in both z_1, z_2; the corresponding selfdual field strength will be analytic as well. The form of U_2 exactly mimics the undeformed one of ref. [23, 36]. Of course, for this to make sense, we have to further enlarge the algebras adding as a generator $1/|y_2|$ with consistent commutation relations; this is done in appendix A.1 of [21]. By generalization of the undeformed reults [23, 36], we are led to the

Conjecture. Performing the singular gauge transformation U_n recursively defined by $U_0 = \mathbf{1}_2$ and

$$U_n \equiv U_n(z_1, \ldots, z_n) := U_{n\text{-}1}(z_1, \ldots, z_{n\text{-}1}) U_{n\text{-}1}^{-1}(y) \frac{\bar{z}_n}{|z_n|}, \tag{7.10}$$

with $U_m(y)$ the function of $y_1, \ldots y_m$ only defined by $U_m(y) := U_m(z_1 - z_n, \ldots, z_{n\text{-}1} - z_n)$, we finally obtain a regular n-istanton solution

$$A \equiv A_n = U_n^{-1} \left(\hat{A} U_n + dU_n \right) \tag{7.11}$$

and a corresponding regular selfdual field strength, for any n.

Results for the **n-antiinstanton solutions** are obtained by the already mentioned replacements. In particular, the singular ones $\hat{A}$ are simply obtained replacing $\hat{\mathcal{D}}$ with $\hat{\mathcal{D}}'$ in (5.9).

References

[1] M. F. Atiyah, *Geometry on Yang-Mills fields*, Lezioni fermiane, Scuola Normale Superiore di Pisa (1979).

[2] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[3] P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml and J. Wess, *Twisted gauge theories*, *Lett. Math. Phys.* **78** (2006) 61 [hep-th/0603024].

[4] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett.* **B 59** (1975) 85.

[5] F. Bonechi, N. Ciccoli and M. Tarlini, *Noncommutative instantons on the* 4*-sphere from quantum groups*, *Commun. Math. Phys.* **226** (2002) 419 [math.QA/0012236].

[6] U. Carow-Watamura, M. Schlieker, S. Watamura, $\mathrm{SO}_q(N)$ *covariant differential calculus on quantum space and quantum deformation of Schrödinger equation*, *Z. Physik* **C 49** (1991) 439.

[7] B.L. Cerchiai, G. Fiore and J. Madore, *Geometrical tools for quantum euclidean spaces*, *Commun. Math. Phys.* **217** (2001) 521 [math.QA/0002007].

[8] A. Connes, *Noncommutative geometry*, Academic Press (1994).

[9] A. Connes, *Non-commutative geometry and physics*, Les Houches, Session LVII, Elsevier Science B. V. (1994).

[10] A. Connes and G. Landi, *Noncommutative manifolds: the instanton algebra and isospectral deformations*, *Commun. Math. Phys.* **221** (2001) 141 [math.QA/0011194].

[11] L. Dabrowski, G. Landi and T. Masuda, *Instantons on the quantum 4-spheres S_q^4*, *Commun. Math. Phys.* **221** (2001) 161 [math.QA/0012103].

[12] M. Dimitrijevic, F. Meyer, L. Moller and J. Wess, *Gauge theories on the Kappa-Minkowski spacetime*, *Eur. Phys. J.* **C 36** (2004) 117 [hep-th/0310116].

[13] V. Drinfeld, *Quantum groups* in I.C.M. Proceedings, Berkeley, p. 798 (1986).

[14] L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtadjan, *Quantization of Lie groups and Lie algebras*, *Algebra i Analyz* **1** (1989) 178, translated from the Russian in *Leningrad Math. J.* **1** (1990) 193.

[15] L.D. Faddeev, L.A. Takhtadjan, *Liouville model on the lattice*, Lecture Notes in Physics, **246**, Springer, New York, p. 166 (1989).

[16] J.M. Gracia-Bonda, J.C. Varilly and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser Boston, Boston (2001).

[17] G. Fiore, *Quantum Groups* $\mathrm{SO}_q(N), Sp_q(n)$ *have q-determinants, too*, *J. Phys.* **A 27** (1994) 3795.

[18] G. Fiore, *Quantum group covariant (anti)symmetrizers, ε-tensors, vielbein, Hodge map and laplacian*, *J. Phys.* **A 37** (2004) 9175.

[19] G. Fiore, *On the hermiticity of q-differential operators and forms on the quantum Euclidean spaces $\mathbb{R}_q^N$*, *Rev. Math. Phys.* **18** (2006) 79 [math.QA/0403463].

[20] G. Fiore, *On q-quaternions*, in preparation.

[21] G. Fiore, *q-quaternions and q-deformed* SU(2) *instantons*, hep-th/0603138.

[22] G. Fiore, *q-deformed quaternions and* SU(2) *instantons*, Preprint 06- Dip. Matematica e Applicazioni, Università di Napoli, and DSF/17-2006. To appear in the proceedings of the *Noncommutative Geometry in Field and String Theories*, Satellite Workshop of "CORFU Summer Institute 2005".

[23] J.J. Giambiagi and K.D. Rothe, *Regular n instanton fields and singular gauge transformations*, *Nucl. Phys.* **B 129** (1977) 111.

[24] T.A. Ivanova, O. Lechtenfeld and H. Muller-Ebhardt, *Noncommutative moduli for multi-instantons*, *Mod. Phys. Lett.* **A 19** (2004) 2419 [hep-th/0404127].

[25] R. Jackiw and C. Rebbi, *Conformal properties of a Yang-Mills pseudoparticle*, *Phys. Rev.* **D 14** (1976) 517.

[26] B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, *Construction of non-abelian gauge theories on noncommutative spaces*, *Eur. Phys. J.* **C 21** (2001) 383 [hep-th/0104153].

[27] G. Landi, *An introduction to noncommutative spaces and their geometries*, Springer-Verlag, Berlin (1997).

[28] G. Landi and W. van Suijlekom, *Noncommutative instantons from twisted conformal symmetries*, math.QA/70601554.

[29] J. Madore, *An introduction to noncommutative differential geometry and its physical applications*, Second edition, London Mathematical Society Lecture Note Series, **257**, Cambridge University Press, Cambridge (1999).

[30] S. Majid, *Braided momentum in the q-Poincaré group*, *J. Math. Phys.* **34** (1993) 2045 [hep-th/9210141].

[31] S. Majid, *q-euclidean space and quantum group wick rotation by twisting*, *J. Math. Phys.* **35** (1994) 5025 [hep-th/9401112].

[32] For a review see for instance: S. Majid, *Foundations of quantum groups*, Cambridge Univ. Press (1995) and references therein.

[33] N. Nekrasov, A. Schwarz, *Instantons on noncommutative* $\mathbb{R}^4$*, and* $(2,0)$ *superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689.

[34] O. Ogievetsky, *Differential operators on quantum spaces for* $\mathrm{GL}_q(n)$ *and* $\mathrm{SO}_q(n)$, *Lett. Math. Phys.* **24** (1992) 245.

[35] O. Ogievetsky and B. Zumino, *Reality in the differential calculus on q euclidean spaces*, *Lett. Math. Phys.* **25** (1992) 121 [hep-th/9205003].

[36] D.I. Olive, S. Sciuto, R.J. Crewther, *Instantons in field theory*, *Riv. Nuovo Cim.* **2** (1979) 1.

[37] P. Podles and S.L. Woronowicz, *Quantum deformation of Lorentz group*, *Commun. Math. Phys.* **130** (1990) 381.

[38] A. Schirrmacher, *Remarks on the use of R-matrices*, in *Quantum groups and related topics*, Kluwer Acad. Publ., Dordrecht, p.55 (1992);
A. Sudbery, *Canonical differential calculus on quantum general linear groups and supergroups*, *Phys. Lett.* **B 284** (1992) 61; , *Phys. Lett.* **B 291** (1992) 519.

[39] P. Schupp, P. Watts, B. Zumino, *Differential geometry on linear quantum groups*, *Lett. Math. Phys.* **25** (1992) 139.

[40] M. Schlieker, W. Weich and R. Weixler, *Inhomogeneous quantum groups*, *Z. Physik* **C 53** (1992) 79.

[41] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[42] R.G. Swan, *Vector bundles and projective modules*, *Trans. Am. Math. Soc.* **105** (1962) 264.

[43] H. Steinacker, *Integration on quantum euclidean space and sphere in* N *dimensions*, *J. Math. Phys.* **37** (1996) 4738.

[44] G. 't Hooft, *Computation of the quantum effects due to a four-dimensional pseudoparticle*, *Phys. Rev.* **D 14** (1976) 3432.

[45] S.L. Woronowicz, *Compact matrix pseudogroups*, *Commun. Math. Phys.* **111** (1987) 613.

[46] S. L. Woronowicz, *Twisted* $\mathrm{SU}(2)$ *group, an example of a non-commutative differential calculus*, Publ. RIMS, Kyoto Univ. **23** (1987) 117.

[47] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*, *Commun. Math. Phys.* **122** (1989) 125.

[48] F. Wilczeck, in *Quark confinement and field theory*, Ed. D. Stump and D. Weingarten, John Wiley and Sons, New York (1977);
E. Corrigan and D.B. Fairlie, *Scalar field theory and exact solutions to a classical* $\mathrm{SU}(2)$ *gauge theory*, *Phys. Lett.* **B 67** (1977) 69;
R. Jackiw, C. Nohl and C. Rebbi, *Conformal properties of pseudoparticle configurations*, *Phys. Rev.* **D 15** (1977) 1642.